JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 27, No. 4, July–August 2004

Solving Relative Two-Point Boundary Value Problems: Spacecraft Formation Flight Transfers Application

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The dynamics of reconfiguration of a formation of spacecraft is studied. The problem is posed as a two-point boundary value problem where the initial and final positions and times are known, and the necessary velocities for the reconfiguration transfers must be found. An algorithm is presented that solves this problem for reconfiguration about a nominal trajectory in arbitrary gravity fields, providing a nonlinear analytical solution to this problem in the spatial domain. The technique being developed is based on a novel application of Hamiltonian dynamics, generating functions, canonical transformations, and the Hamilton–Jacobi equation. A specific example of reconfiguration of spacecraft about an Earth–sun libration point is studied that shows that the optimal transfer time depends strongly on the initial and final positions.

Introduction

THE potential benefits of missions involving a large number of spacecraft have led to great current interest in the design of multispacecraft missions. One benefit of such missions is the possibility of reconfiguring the spacecraft to address changing mission goals. However, each reconfiguration of the formation has associated fuel costs and, thus, reduces the life-time of the mission. This specific problem is an active area of research, and optimal solutions have been found for specific changes in formation configuration. Zhan et al.[1] studied the dynamics of a formation of spacecraft on elliptical orbits around the Earth, moving relative to a reference spacecraft on a circular orbit. A similar study was achieved in the Hill's three-body problem by Richardson and Mitchell,[2] the reference spacecraft being at one of the libration points. These studies assume a spherical Earth and circular reference orbits. Alfriend et al.[3] proposed a new approach that includes nonlinear perturbation, namely, the reference orbit eccentricity and J_2 effects. The osculating elements are obtained from the mean elements for each spacecraft in the formation and then transformed to a rotating reference frame centered at the "Chief" to view the relative motion. Albeit efficient for specific situations, these methods fail to provide a general solution to the reconfiguration problem involving an arbitrary number of spacecraft, a more complex dynamic environment, or more complicated formation configurations. The goal of this paper is twofold. First, we provide a method that solves the reconfiguration problem for an arbitrary number of spacecraft in a general Hamiltonian dynamic environment by the use of impulsive transfers. Second, we develop techniques that make this method numerically tractable for a class of problems. Our method is numerically independent of the complexity of the configuration, of the dynamic environment, and of the number of spacecraft. In the future, our method can be combined with the method proposed by Wang and Hadaegh[4] to solve the minimum-fuel formation reconfiguration problem in realistic dynamic scenarios.

Our approach uses the generating functions for canonical transformations in Hamiltonian dynamic systems to describe the relative nonlinear motion of a formation of spacecraft as the solution to a two-point boundary value problem. By doing so, we can specify the initial and final configuration and immediately know what the corresponding velocities should be to carry out the reconfiguration in a given time. Our approach is distinct from the classical use of generating functions; we are using them to solve the boundary value problem, whereas in previous applications they were used to solve the initial value problem.[5,6] One consequence of this is that we must solve for the generating functions explicitly instead of evaluating their partial derivatives only. Our approach is similar, in a sense, to the one used in applied mathematics to build symplectic integrators.[7]

This paper is organized into three main parts: First, we explain how a reconfiguration problem can be solved by the use of generating functions, next we provide an algorithm that solves the problem, and, finally, some examples are given for the reconfiguration of spacecraft about an Earth–sun libration point.

Hamilton–Jacobi Theory

In this section, we briefly show how the generating functions are derived from the variational principle. This provides the necessary background for the method we have developed. For more details on generating functions and how they can be derived by the use of symplectic geometry, we refer the reader to Refs. 5, 6, and 8–12.

Let H be the Hamiltonian function of a Hamiltonian dynamic system with n degrees of freedom. Then the equations of motion can be derived from the principle of least action:

The integral $\int p dq - H\,dt$ has a critical point in the class of curves γ, whose ends lie in the n-dimensional subspaces ($t = t_0$, $q = q_0$, p_0 free), and ($t = t_1$, $q = q_1$, p_1 free), of extended phase space:

$$\delta \int_\gamma (p\dot{q} - H)\,dt = \int_\gamma \left(\dot{q}\delta p + p\delta\dot{q} - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p\right) dt$$
$$= [p\delta q]_0^1 + \int_\gamma \left[\left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p - \left(\dot{p} + \frac{\partial H}{\partial q}\right)\delta q\right] dt = 0 \tag{1}$$

Therefore, because the variation vanishes at the end points, we get

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{2}$$

Equations (2) are Hamilton's equations. We point out that the principle of least action is coordinate invariant, a property we will use later.

Now consider a contact transformation ϕ, that is, a transformation in phase space from one set of coordinates (q_i, p_i) to a new set

Received 10 October 2002; revision received 25 May 2003; accepted for publication 13 November 2003.

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(Q_i, P_i) defined by $2n$ equations of the form

$$(Q_i, P_i) = \phi_i(q, p, t) \tag{3}$$

In Eq. (3), ϕ is said to be a canonical transformation if it preserves Hamilton's equations. (Authors disagree on the exact definition of a canonical transformation. Our definition differs from the one of Arnold,[9] but is in agreement with Abraham,[8] Godstein,[5] and Greenwood.[6] Arnold's definition considers the canonical transformations as being a subclass of maps preserving Hamilton's equations.) Hence, the assumption that ϕ is canonical yields

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{4}$$

where $K = K(Q, P, t)$ is the Hamiltonian of the system expressed as a function of the new coordinates. To find the relation between K and H, we use the principle of least action, which holds for both sets of coordinates (as noted before):

$$\delta \int_{t_0}^{t_1} \left[\sum_i p_i \dot{q}_i - H(q, p, t) \right] \mathrm{d}t = 0 \tag{5}$$

$$\delta \int_{t_0}^{t_1} \left[\sum_i P_i \dot{Q}_i - K(Q, P, t) \right] \mathrm{d}t = 0 \tag{6}$$

From Eqs. (5) and (6), we conclude that the integrands of the two integrals differ at most by a total time derivative of an arbitrary function F:

$$\sum_i p_i \dot{q}_i - H \sum_j P_j \dot{Q}_j - K + \frac{\mathrm{d}}{\mathrm{d}t} F \tag{7}$$

Such a function is called a generating function of the transformation and is a priori a function of both the old and new variables and time. However, the two sets of coordinates are connected by the $2n$ equations in Eq. (3). Thus, we conclude that F is a function of the $2n + 1$ independent variables only. In general, F depends on n old coordinates and n new coordinates. Hence, the generating functions can be written as one of the functions:

$$F_1(q, Q, t), \qquad F_2(q, P, t), \qquad F_3(p, Q, t), \qquad F_4(p, P, t) \tag{8}$$

If q and Q are the two independent variables relevant for a given problem, then F_1 will be used. Expanding the total time derivative of F_1 yields

$$\frac{\mathrm{d}}{\mathrm{d}t} F_1(q, Q, t) = \sum_i \frac{\partial F_1}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F_1}{\partial t} \tag{9}$$

Then Eq. (7) reads

$$\sum_i \left(p_i - \frac{\partial F_1}{\partial q_i} \right) \dot{q}_i - H = \sum_j \left(P_j + \frac{\partial F_1}{\partial Q_j} \right) \dot{Q}_j - K + \frac{\partial F_1}{\partial t} \tag{10}$$

Under the assumption that q and Q are independent, Eq. (10) is equivalent to

$$p_i = \frac{\partial F_1}{\partial q_i} \tag{11}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} \tag{12}$$

$$K = H + \frac{\partial F_1}{\partial t} \tag{13}$$

If q and P are independent variables, then we may use the generating function F_2. We rewrite Eq. (7) as a function of the two independent variables q and P:

$$\sum_i p_i \dot{q}_i - H = -\sum_j Q_j \dot{P}_j - K + \frac{\mathrm{d}}{\mathrm{d}t} F_2 \tag{14}$$

where

$$F_2(q, P, t) = F_1(q, Q, t) + \sum_i P_i Q_i \tag{15}$$

Equation (12) is used to express the Q as a function of the q and P in Eq. (15). Equation (15) is called a Legendre transformation[5,6,10,13] and can be used to transform between F_1 and F_2.

We obtain

$$p_i = \frac{\partial F_2}{\partial q_i} \tag{16}$$

$$Q_i = \frac{\partial F_2}{\partial P_i} \tag{17}$$

$$K = H + \frac{\partial F_2}{\partial t} \tag{18}$$

The same reasoning can be applied for F_3 and F_4, if (p, Q) or (p, P) are assumed to be independent variables, to find

$$q_i = -\frac{\partial F_3}{\partial p_i} \tag{19}$$

$$P_i = -\frac{\partial F_3}{\partial Q_i} \tag{20}$$

$$K = H + \frac{\partial F_3}{\partial t} \tag{21}$$

$$q_i = -\frac{\partial F_4}{\partial p_i} \tag{22}$$

$$Q_i = \frac{\partial F_4}{\partial P_i} \tag{23}$$

$$K = H + \frac{\partial F_4}{\partial t} \tag{24}$$

Also, Eq. (15) can be generalized to F_3 and F_4, and is

$$F_3(p, Q, t) = F_1(q, Q, t) - \sum_i q_i p_i$$

$$F_4(p, P, t) = F_1(q, Q, t) - \sum_i P_i Q_i - \sum_i p_i q_i \tag{25}$$

Finally, we choose the new variables to be constants of motion by requiring that the new Hamiltonian K be identically zero. Hamilton's equations become

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i = 0 \tag{26}$$

$$-\frac{\partial K}{\partial Q_i} = \dot{P}_i = 0 \tag{27}$$

and Eq. (13) reads

$$\frac{\partial F_1}{\partial t} + H\left(q, \frac{\partial F_1}{\partial q}, t \right) = 0 \tag{28}$$

Equation (28) is often referred to as the Hamilton–Jacobi equation. F_1 is now a function of the actual position q and constants of motion Q.

The same simplification applies to Eqs. (18), (21), and (24):

$$\frac{\partial F_2}{\partial t} + H\left(q, \frac{\partial F_2}{\partial q}, t\right) = 0 \tag{29}$$

$$\frac{\partial F_3}{\partial t} + H\left(-\frac{\partial F_3}{\partial p}, p, t\right) = 0 \tag{30}$$

$$\frac{\partial F_4}{\partial t} + H\left(-\frac{\partial F_4}{\partial p}, p, t\right) = 0 \tag{31}$$

We note that F_1 and F_2 (F_3 and F_4) verify the same equation.

Now, because the transformation of phase space induced by the phase flow is canonical,[9] it can be represented by generating functions. Previous studies of the phase flow in which generating functions and the Hamilton–Jacobi equation were used often consider the energy (when it is a constant of motion) and the initial time, or ignorable coordinates, as variables of the generating functions. In the present study, we choose (Q, P) to be the initial conditions of the system. Thus, $F_1(q, Q)$ becomes a function of initial and current positions, $F_2(q, P)$ becomes a function of initial momentum and current position, $F_3(p, Q)$ becomes a function of initial position and current momentum and $F_4(p, P)$ becomes a function of initial and current momenta. By the choice of (Q, P) as initial conditions, the Hamilton–Jacobi equation and F_i describe the phase flow as a boundary value problem.

To solve the Hamilton–Jacobi equation for the generating functions of the phase flow, we need to specify initial or boundary values of the function. At $t = 0$, the current position and momentum are the same as the initial position and momentum; therefore, the phase flow induces the identity transformation. Such a transformation can be described by F_2 and F_3 but not by F_1 and F_4 (Refs. 5, 6, and 10).

To understand why F_1 and F_4 cannot describe the identity transformation, we must realize that knowledge of Q and $q = Q$ is not sufficient to determine P and $p = P$ uniquely. Thus, the boundary problem that consists of finding the initial and final momenta given the initial and final positions is ill-posed at $t = 0$, and F_1 and F_4 are not well-defined functions at the initial time. We say that they develop singularities at the initial time. To bypass this problem, we must use the Legendre transformation to compute their values at some later epoch to integrate the Hamilton–Jacobi equation [Eqs. (28) and (31)]. The singularity in F_1 and F_4 arises from a nonuniqueness of the boundary value problem, that is, there are an infinite number of solutions that satisfy $q = Q$ at $t = t_0$.

In the preceding text we have developed a local theory that guarantees local existence of the generating functions, but we have not said anything about global existence. In general, we do not know, a priori, if the generating functions will be defined for all time, and in most cases we find that they develop singularities at specific times. It can be proven, however, that they cannot all be singular at the same time.[9] (Actually, one can define 2^n generating functions. Arnold[9] proved that among them, one, at least, must be well defined. In practice, among the four generating functions we defined, one is always nonsingular.) More details on singularities can be found in Refs. 9 and 14–16. Global singularities again correspond to boundary value problems that have a multiple or infinite number of solutions. In general, computation of an F_i cannot be continued through a singularity, although the F_i are well defined on either side of the singularity. This is the situation that occurs, for example, in the solution of Lambert's problem over a 180-deg transfer: There are an infinite number of transfers that satisfy the boundary value problem (occurring at different orbit planes).

Resolution of the Boundary Value Problem Using the Hamilton–Jacobi Equation

Using the theory developed in the preceding section, we are now able to solve, in principle, any boundary value problem of a Hamiltonian system. Consider the change of configuration of n spacecraft moving in formation. Initially, spacecraft i has position Q^i in the formation, and it is desired to move it to position q^i in T units of time. To solve this, we need to solve a boundary value problem in terms of initial and final position (a Lambert's type of problem). The procedure will be as follows:

1) Solve the Hamilton–Jacobi equation (28) to find a closed-form solution for $F_1(q, Q, t)$.

2) Solve the boundary value problem with Eqs. (11) and (12). The initial momentum is given by

$$P^i = -\frac{\partial F_1}{\partial Q}(q^i, Q^i, T) \tag{32}$$

and the final momentum is

$$p^i = \frac{\partial F_1}{\partial q}(q^i, Q^i, T) \tag{33}$$

Then we find a solution to the boundary value problem assuming impulsive maneuvers. The change of configuration proceeds as follows: Apply an impulse to change the initial momentum to P^i; let the spacecraft trajectory evolve; and, at time $t = T$, when the spacecraft has reached its final position, apply another impulse to change the momentum from p^i to its desired value. In general, the spacecraft will have some desired initial and final momenta, but, without loss of generality, we neglect this detail in our current work.

Given F_i and its partial derivatives, we only need to perform two function evaluations to solve the boundary value problem [Eqs. (32) and (33)]. Hence, for a formation of N spacecraft, we only need to perform $2N$ function evaluations. This makes our method especially valuable for a large collection of spacecraft because $N!$ boundary value problems must be solved to fully characterize the transfer of N spacecraft in distinct initial positions to N new positions.

In this example, we assume that a closed-form solution to the Hamilton–Jacobi equation can be found, which is true for very few Hamiltonian systems. The harmonic oscillator is one such example, and we will study it in the next section to illustrate our method. Then we will show that in the particular case of formation flight, under some spatial limitations, we can find approximate solutions to the Hamilton–Jacobi equation in closed form to as high an accuracy as we want.

Given such a closed-form solution for the boundary value problem, many applications exist. One such application allows us to determine directly the optimal set of transfers for a formation of spacecraft. The optimal transfer for spacecraft with impulsive maneuvers consists of the design of a final configuration that meets various mission-dependent constraints while minimizing a cost function. (Generally, there may be k relationships constraining the final relative positions of the spacecraft.) Typically, the cost function depends on the initial and final impulse needed to achieve the transfer and/or the time transfer. By the use of generating functions, the cost function can be expressed as a function of the transfer time and the initial and final configuration. If the initial configuration is given, the cost function becomes a function of the final configuration and transfer time only. The optimal transfer design problem is then reduced to minimization of the new cost function with respect to $n + 1$ variables with at most k constraints. If these constraints can be solved analytically, then the problem becomes an optimization problem of an analytic function expressed in a closed form with respect to $n + 1 - k$ variables. We present a simple example of this later, and we will see that such a problem can be handled graphically.

Boundary Value Problem for a One-Dimensional Harmonic Oscillator

We now illustrate our approach with a simple example, the harmonic oscillator. We solve two different kinds of boundary value problems:

1) Given an initial momentum P, final position q, and time interval T, what is the initial position Q and final momentum p?

2) Given an initial and final position Q and q and a time interval T, what is the corresponding initial and final momenta P and p?

In the first problem (q, P) are given, hence, F_2 is the relevant generating functions to use. For the second one, (q, Q) are given and we should use F_1.

The Hamiltonian of the harmonic oscillator is

$$H(q, p) = (1/2m)p^2 + (k/2)q^2 \tag{34}$$

Thus, F_2 must satisfy Eq. (29):

$$\frac{\partial F_2}{\partial t} + \frac{1}{2m}\left(\frac{\partial F_2}{\partial q}\right)^2 + \frac{k}{2}q^2 = 0 \tag{35}$$

Because H is a second-order polynomial in q and p, it is reasonable to look for F_2 as a polynomial in q and P:

$$F_2(q, P, t) = \sum_{i,j} a_{i,j}(t) q^i P^j \tag{36}$$

Now substitute Eq. (36) into Eq. (35) and solve for each coefficient of F_2 by the use of the initial conditions $t = 0$, where F_2 is the identity transformation $F_2(q, P, 0) = qP$: $a_{11}(0) = 1$, $a_{i,j}(0) = 0$ for all $(i, j) \neq (1, 1)$. This yields

$$F_2 = -\tfrac{1}{2}\sqrt{km}\tan\left(\sqrt{k/m}\,t\right)q^2 + \sec\left(\sqrt{k/m}\,t\right)qP - \tfrac{1}{2}\left(1\big/\sqrt{km}\right)\tan\left(\sqrt{k/m}\,t\right)P^2 \tag{37}$$

Then, Eqs. (16) and (17) allow us to solve the boundary problem:

$$p = \frac{\partial F_2}{\partial q} = -\sqrt{km}\tan\left(\sqrt{\frac{k}{m}}t\right)q + \sec\left(\sqrt{\frac{k}{m}}t\right)P$$

$$Q = \frac{\partial F_2}{\partial P} = \sec\left(\sqrt{\frac{k}{m}}t\right)q - \frac{1}{\sqrt{km}}\tan\left(\sqrt{\frac{k}{m}}t\right)P \tag{38}$$

Once the generating functions are found, it is clear that we can arbitrarily solve boundary value problems simply by evaluating Eq. (38).

To solve the second boundary value problem, we cannot use the same approach we used for F_2 because F_1 cannot generate the identity transformation at $t = 0$. Instead, we use the Legendre transformation defined by Eq. (15) to obtain F_1 from our known solution of F_2. First, using Eq. (38), we find P as a function of (Q, q):

$$P = \sqrt{km}\left[\sec\left(\sqrt{k/m}\,t\right)\cot\left(\sqrt{k/m}\,t\right)q - \cot\left(\sqrt{k/m}\,t\right)Q\right] \tag{39}$$

then, replacing P in Eq. (15) yields

$$F_1 = \tfrac{1}{2}\sqrt{km}\csc\left(\sqrt{k/m}\,t\right)\left[-2qQ + (q^2 + Q^2)\cos\left(\sqrt{k/m}\,t\right)\right. \tag{40}$$

We notice that F_1 is singular at times $t = \sqrt{(m/k)}n\pi$, $n \in \mathbf{Z}$, and that F_2 is singular at $t = \sqrt{(m/k)}[(2n+1)\pi/2]$, $n \in \mathbf{Z}$. This illustrates our earlier remarks on singularities. First, both generating functions are singular for different times. Moreover, the singularities correspond to an ill-posed boundary value problem. If $t = \sqrt{(m/k)}n\pi$, then, from the equations of motion of the harmonic oscillator, we obtain

$$q\left(\sqrt{m/k}n\pi\right) = \pm Q \tag{41}$$

$$p\left(\sqrt{m/k}n\pi\right) = \pm P \tag{42}$$

Hence, given (q, Q), it is clear that we cannot solve for (p, P) uniquely. Indeed, there are infinitely many solutions. However, at the same time, F_2 yields $p = \pm P$ and $Q = \pm q$ and is well defined. The issue of the generating functions becoming singular is an important one in the present study, and we will come back to it later in more detail. As illustrated here and mentioned by Arnold,[14] even for very simple systems singularities appear.

Relative Two-Point Boundary Value Problem

Formulation of the Problem in Terms of Hamiltonian Equations

We have shown in the preceding sections how the generating functions solve boundary value problems, and we have illustrated this using the harmonic oscillator. In general, as mentioned before, we are not able to solve the Hamilton–Jacobi equation, and, therefore, we cannot apply the preceding theory. However, to study spacecraft formation trajectories, we only need to study relative states of spacecraft. For this class of problem, the Hamiltonian function has a particular structure that enables us to solve the Hamilton–Jacobi equation, independently of the complexity of the gravitational field. Generating functions found are associated with the phase flow describing relative motion, hence, they solve the relative boundary value problem.

Consider a Hamiltonian system, such as a spacecraft moving under gravitational forces with Hamiltonian function $H(q, p, t)$. We introduce a few notations:

Let (Q_0, P_0) and (Q_1, P_1) be two points in phase space such that

$$Q_1 = Q_0 + \Delta Q \tag{43}$$

$$P_1 = P_0 + \Delta P \tag{44}$$

where $(\Delta Q, \Delta P)$ are small in a sense we will define later. We denote by (q_i, p_i) the trajectory with initial conditions (Q_i, P_i) that is,

$$q_1 = q(Q_1, P_1, t) \qquad p_1 = p(Q_1, P_1, t) \tag{45}$$

$$q_0 = q(Q_0, P_0, t) \qquad p_0 = p(Q_0, P_0, t) \tag{46}$$

We define

$$X_h = \begin{pmatrix} \Delta q \\ \Delta p \end{pmatrix}$$

by

$$X_1 = X_0 + X_h \tag{47}$$

where

$$X_i = \begin{pmatrix} q_i \\ p_i \end{pmatrix}$$

For convenience, we shall call (q_0, p_0) the nominal trajectory and (q_1, p_1) the displaced trajectory.

Both trajectories verify Hamilton's equations:

$$\dot{X}_i = J\nabla H_i \tag{48}$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \nabla H_i = \begin{pmatrix} \dfrac{\partial H}{\partial q} \\[2ex] \dfrac{\partial H}{\partial p} \end{pmatrix}(q_i, p_i, t)$$

By the use of our earlier notation, Eq. (48) reads, for $i = 1$,

$$\dot{X}_0 + \dot{X}_h = J\nabla H_1 \tag{49}$$

We expand the right-hand side of Eq. (49) about the nominal trajectory X_0, assuming $(\Delta q, \Delta p)$ small enough for convergence of the series:

$$\nabla H(q_1, p_1, t) = \nabla H(q_0, p_0, t) + \begin{pmatrix} \dfrac{\partial^2 H}{\partial q^2}(q_0, p_0, t)\Delta q + \dfrac{\partial^2 H}{\partial q \partial p}(q_0, p_0, t)\Delta p \\[2ex] \dfrac{\partial^2 H}{\partial q \partial p}(q_0, p_0, t)\Delta q + \dfrac{\partial^2 H}{\partial p^2}(q_0, p_0, t)\Delta p \end{pmatrix} + \cdots \tag{50}$$

Substitution of this into Eq. (49) yields

$$\dot{X}_0 + \dot{X}_h = J\nabla H_0 + J\begin{pmatrix} \dfrac{\partial^2 H}{\partial q^2}(q_0, p_0, t)\Delta q + \dfrac{\partial^2 H}{\partial q \partial p}(q_0, p_0, t)\Delta p \\ \dfrac{\partial^2 H}{\partial q \partial p}(q_0, p_0, t)\Delta q + \dfrac{\partial^2 H}{\partial p^2}(q_0, p_0, t)\Delta p \end{pmatrix} + \cdots \tag{51}$$

By the use of Eq. (48), Eq. (51) simplifies to

$$\dot{X}_h = J\begin{pmatrix} \dfrac{\partial^2 H}{\partial q^2}(q_0, p_0, t)\Delta q + \dfrac{\partial^2 H}{\partial q \partial p}(q_0, p_0, t)\Delta p \\ \dfrac{\partial^2 H}{\partial q \partial p}(q_0, p_0, t)\Delta q + \dfrac{\partial^2 H}{\partial p^2}(q_0, p_0, t)\Delta p \end{pmatrix} + \cdots \tag{52}$$

This is a Hamiltonian system if and only if there exists an Hamiltonian function H_h such that Eq. (52) can be written as Hamilton's equations. Let

$$H_h(X_h, t) = \frac{1}{2}X_h \begin{pmatrix} \dfrac{\partial^2 H}{\partial q^2}(q_0, p_0, t) & \dfrac{\partial^2 H}{\partial q \partial p}(q_0, p_0, t) \\ \dfrac{\partial^2 H}{\partial q \partial p}(q_0, p_0, t) & \dfrac{\partial^2 H}{\partial q^2}(q_0, p_0, t) \end{pmatrix} X_h + \cdots \tag{53}$$

We can check that

$$\dot{X}_h = J\nabla H_h(X_h, t) \tag{54}$$

Without ignoring higher-order terms, the expansion of the right-hand side of Eq. (49) yields

$$H_h(X_h, t) = \sum^{n} \frac{\Delta q^i \Delta P^j}{i!j!} \frac{\partial^{i+j} H}{\partial q^i \partial P^j}(q_0, p_0, t) \qquad i, j = 0, \qquad i + j \geq 2 \tag{55}$$

Thus, the dynamics of a particle relative to a known trajectory is Hamiltonian with a Hamiltonian function $H_h(X_h, t) = H_h(\Delta q, \Delta p, t)$. The coefficients of the Taylor series

$$\frac{1}{i!j!}\frac{\partial^{i+j} H}{\partial q^i \partial P^j}(q_0, p_0, t)$$

are time-varying quantities and are easily evaluated for any Hamiltonian.

Hamilton–Jacobi Equations

In the preceding section we found that the Hamiltonian describing the dynamics of two particles relative to each other is a power series in its spatial variables $(\Delta q, \Delta p)$, with time-dependent coefficients. At first glance, the solution of the Hamilton–Jacobi equation for a Hamiltonian system whose Hamiltonian function has infinitely many terms may appear impractical. However, if we truncate H_h, a closed-form solution for the generating functions can be found.

The procedure to solve the Hamilton–Jacobi equation using a truncated H_h is similar in nature to the one we used for the harmonic oscillator. (The harmonic oscillator is a particular case where only the first term in the series expansion is nonzero.) Let us sketch out the main ideas. We truncate the series H_h to keep finitely many terms only. Suppose N terms are kept, then we say that we describe the relative motion using an approximation of order N. Clearly, the greater N is, the better is our approximation to the nonlinear motion of a particle about the nominal trajectory. When an approximation of order N is used, we look for a generating function F_i as a polynomial of order N in its spatial variables with time-dependent coefficients. Then, the Hamilton–Jacobi equation (28) can be viewed as being of the form

$$P(Y, t) = 0 \quad \forall Y \tag{56}$$

where Y is a vector containing the spatial variables of the generating function (typically, $Y = (\Delta q, \Delta Q)$ for F_1, $Y = (\Delta q, \Delta P)$ for F_2, etc.), and P is polynomial of order N in Y with time-dependent coefficients. (Terms of order more than N are irrelevant because we have already neglected some when we truncated H_h.) The coefficients of P are a combination of the coefficients of F_i and their time derivatives. Equation (56) holds for all Y if and only if all of the coefficients of P are zero. In this manner, we define a set of ordinary differential equations whose solution is the coefficients of the generating function F_i up to order N.

The procedure of our algorithm is summarized as follows:

1) Approximate the Hamiltonian describing the relative motion by truncation of high-order terms.

2) Assume that the generating function is a polynomial in its spatial variables with time-dependent coefficients, that is, a truncated Taylor series expansion about the nominal trajectory.

3) Transform the partial differential Hamilton–Jacobi equation into a system of ordinary differential equations by balancing terms of the same order in the Taylor series expansion.

4) Solve the system of ordinary differential equations.

We now proceed in detail. First we note that F_i has no coefficient at the first order. Indeed, suppose that $F_2 = a(t)\Delta q + b(t)\Delta P$, then $\Delta q = \partial F_2/\partial \Delta p = a(t)$ and $\Delta Q = b(t)$. However, $(\Delta q, \Delta p, \Delta Q, \Delta P)$ are assumed to be small at first order at least, and so we conclude that $a(t) = b(t) = 0$. Thus, at leading order, F_i is a quadratic form in its two variables, without a linear term, and H_h satisfies Eq. (53).

Restricting ourselves for the moment to second order, define H_h and F_2 by the use of block matrices:

$$H_h = \frac{1}{2}X_h^T \begin{pmatrix} H_{qq}(t) & H_{qp}(t) \\ H_{pq}(t) & H_{pp}(t) \end{pmatrix} X_h \tag{57}$$

$$F_2 = \frac{1}{2}Y^T \begin{pmatrix} F_{11}^2(t) & F_{12}^2(t) \\ F_{21}^2(t) & F_{22}^2(t) \end{pmatrix} Y \tag{58}$$

where

$$Y = \begin{pmatrix} \Delta q \\ \Delta P \end{pmatrix}$$

and the matrix defining H_h and F_2 are both symmetric. The explicit form of H_h is given by Eq. (53). Then, by the use of Eq. (16)

$$\Delta p = \frac{\partial F_2}{\partial \Delta q} = \begin{pmatrix} F_{11}^2(t) & F_{12}^2(t) \end{pmatrix} Y \tag{59}$$

Substitution into Eq. (29) yields

$$0 = Y^T \left\{ \begin{pmatrix} \dot{F}_{11}^2(t) & \dot{F}_{12}^2(t) \\ \dot{F}_{12}^2(t)^T & \dot{F}_{22}^2(t) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} I & F_{11}^2(t)^T \\ 0 & F_{12}^2(t)^T \end{pmatrix} \begin{pmatrix} H_{qq}(t) & H_{qp}(t) \\ H_{pq}(t) & H_{pp}(t) \end{pmatrix} \begin{pmatrix} I & 0 \\ F_{11}^2(t) & F_{12}^2(t) \end{pmatrix} \right\} Y \tag{60}$$

Though the preceding equations have been derived by the use of F_2, they are also valid for F_1 if

$$Y = \begin{pmatrix} \Delta q \\ \Delta P \end{pmatrix}$$

is replaced with

$$Y = \begin{pmatrix} \Delta q \\ \Delta Q \end{pmatrix}$$

because we noticed before that F_1 and F_2 solve the same Hamilton–Jacobi equation. Equation (60) is equivalent to the following four matrix equations:

$$\dot{F}_{11}^{1,2}(t) + H_{qq}(t) + H_{qp}(t)F_{11}^{1,2}(t) + F_{11}^{1,2}(t)H_{pq}(t) + F_{11}^{1,2}(t)H_{pp}(t)F_{11}^{1,2}(t) = 0$$

$$\dot{F}_{12}^{1,2}(t) + H_{qp}(t)F_{12}^{1,2}(t) + F_{11}^{1,2}(t)H_{pp}(t)F_{12}^{1,2}(t) = 0$$

$$\dot{F}_{21}^{1,2}(t) + F_{21}^{1,2}(t)H_{pq}(t) + F_{21}^{1,2}(t)H_{pp}(t)F_{11}^{1,2}(t) = 0$$

$$\dot{F}_{22}^{1,2}(t) + F_{21}^{1,2}(t)H_{pp}(t)F_{12}^{1,2}(t) = 0 \tag{61}$$

where we replaced F_{ij}^2 by $F_{ij}^{1,2}$ to signify that these equations are valid for both F_1 and F_2. Also recall that $F_{21}^{1,2} = F_{12}^{1,2T}$. A similar set of equations can be derived for F_3 and F_4:

$$\dot{F}_{11}^{3,4}(t) + H_{pp}(t) - H_{pq}(t)F_{11}^{3,4}(t) - F_{11}^{3,4}(t)H_{qp}(t) + F_{11}^{3,4}(t)H_{qq}(t)F_{11}^{3,4}(t) = 0$$

$$\dot{F}_{12}^{3,4}(t) - H_{pq}(t)F_{12}^{3,4}(t) + F_{11}^{3,4}(t)H_{qq}(t)F_{12}^{3,4}(t) = 0$$

$$\dot{F}_{21}^{3,4}(t) - F_{21}^{3,4}(t)H_{qp}(t) + F_{21}^{3,4}(t)H_{qq}(t)F_{11}^{3,4}(t) = 0$$

$$\dot{F}_{22}^{3,4}(t) + F_{21}^{3,4}(t)(t)H_{qq}(t)F_{12}^{3,4}(t) = 0 \tag{62}$$

Note that the first equations of Eqs. (61) and (62) are Ricatti equations, the second and third are nonhomogeneous, time-varying, linear equations and are equivalent to each other, that is, transform into each other under transpose, and the last are just a quadrature.

If we want to generalize this method to higher order, tensor notation is required. Following are the equations derived for F_2 for a Hamiltonian system of dimension n. The Taylor expansion is now written as

$$f(x,t) = f^0(t) + f^1(t)\cdot\boldsymbol{x} + [f^2(t)\cdot\boldsymbol{x}]\cdot\boldsymbol{x} + \{[f^3(t)\cdot\boldsymbol{x}]\cdot\boldsymbol{x}\}\cdot\boldsymbol{x} + \cdots \tag{63}$$

Application of this formula to $H(\boldsymbol{x},t)$ and to the canonical transformation $F_2 = F(\boldsymbol{y},t)$ yields

$$H(\boldsymbol{x}) = h_{i,j}(t)x_i x_j + h_{i,j,k}(t)x_i x_j x_k + \cdots \tag{64}$$

$$F(\boldsymbol{y}) = f_{i,j}(t)y_i y_j + f_{i,j,k}(t)y_i y_j y_k + \cdots \tag{65}$$

where we assume the summation convention. Let us now express $\boldsymbol{x} = (\Delta q, \Delta p)$ as a function of $\boldsymbol{y} = (\Delta q, \Delta P)$. [We drop the time dependence in the notation, that is, we shall write $h_{i,j}$ instead of $h_{i,j}(t)$.] For all $a \le n$ and $j = n + a$

$$x_a = y_a \tag{66}$$

$$x_j = \frac{\partial F}{\partial y_a} \tag{67}$$

$$= f_{a,k}y_k + f_{k,a}y_k + f_{a,k,l}y_k y_l + f_{k,a,l}y_k y_l + f_{k,l,a}y_k y_l + \cdots \tag{68}$$

where n is the dimension of the space. The Hamilton–Jacobi equation becomes

$$\frac{\partial F}{\partial t} + H = 0 \tag{69}$$

$$\dot{f}_{i,j}y_i y_j + \dot{f}_{i,j,k}y_i y_j y_k + \cdots + h_{i,j}x_i x_j + h_{i,j,k}x_i x_j x_k + \cdots = 0 \tag{70}$$

Replacement of $\boldsymbol{x}$ by $\boldsymbol{y}$ in Eq. (70) by the use of Eq. (68) and preserving only terms of order less than three yields

$$\begin{aligned}
0 = {}& \dot{f}_{i,j}y_i y_j + \dot{f}_{i,j,k}y_i y_j y_k + h_{a,b}y_a y_b + h_{a,b,c}y_a y_b y_c \\
& + (h_{a,n+b} + h_{n+b,a})y_a(f_{b,k}y_k + f_{k,b}y_k + f_{b,k,l}y_k y_l \\
& + f_{l,b,k}y_k y_l + f_{k,l,b}y_k y_l) + h_{n+a,n+b}(f_{a,k}y_k + f_{k,a}y_k \\
& + f_{a,k,l}y_k y_l + f_{l,a,k}y_k y_l + f_{k,l,a}y_k y_l)(f_{b,m}y_m + f_{m,b}y_m \\
& + f_{b,m,p}y_m y_p + f_{p,b,m}y_m y_p + f_{m,p,b}y_m y_p) \\
& + (h_{n+a,b,c} + h_{c,n+a,b} + h_{b,c,n+a})y_b y_c(f_{a,k}y_k + f_{k,a}y_k) \\
& + (h_{n+a,n+b,c} + h_{n+b,c,n+a} + h_{c,n+a,n+b})y_c(f_{a,k}y_k + f_{k,a}y_k) \\
& \times (f_{b,l}y_l + f_{l,b}y_l) + h_{n+a,n+b,n+c}(f_{a,k}y_k + f_{k,a}y_k) \\
& \times (f_{b,l}y_l + f_{l,b}y_l)(f_{c,m}y_m + f_{m,c}y_m)
\end{aligned} \tag{71}$$

Equation (71) is a polynomial equation in the y_i variables with time-dependent coefficients. It holds if every coefficient is zero. We notice that the equations of order two (the one obtained by setting the coefficients of $y_i y_j$ to zero) are the same as the ones found earlier. The equations of order three are given explicitly in Appendices A and B. The process of deriving equations for F can be continued to arbitrarily high-order by the use of a symbolic manipulation program. (We have implemented and solved the expansion to order eight using Mathematica©.)

Now let us look at the initial conditions. We saw earlier that F_2 and F_3 generate the identity transformation at the initial time. They provide multiple ways to generate the identity transformation; however, in our algorithm, we look for F_2 as a polynomial. In that case, F_2 at the initial time must be given by

$$F_2(\Delta q, \Delta P, t=0) = \frac{1}{2}Y^T \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix} Y \tag{72}$$

where I_n is the identity matrix of order n. At initial time, all terms of F_2 but $2n$ are zero, that is, $F_{12}^2(t=0) = F_{21}^2(0) = I_n$, $F_{11}^2(0) = F_{22}^2(0) = 0$, and all higher-order coefficients are zero. The same applies to F_3 and yields

$$F_3(\Delta p, \Delta Q, t=0) = \frac{1}{2}Y^T \begin{pmatrix} 0_n & -I_n \\ -I_n & 0_n \end{pmatrix} Y \tag{73}$$

At initial time, all terms of F_3 but $2n$ are zero, that is, $F_3^{12}(t=0) = F_3^{21}(0) = -I_n$, $F_3^{11}(0) = F_3^{22}(0) = 0$, and all of the higher-order coefficients are zero. F_1 and F_4 have singular coefficients at initial time, and, therefore, we must evaluate them at a later epoch. This is done with Legendre transformations and reversion of series. For these functions, having their coefficients defined at some time t allows us to solve the set of ordinary differential equations for the coefficients forward or backward in time.

Singularities of the Generating Functions

The purpose of this section is to explain how the existence of singularities impacts our algorithm. First recall that whatever system is considered, it is likely that the generating functions will develop singularities at some epoch.[14] These singularities correspond to multiple solutions to the boundary value problem. We saw that F_1 is singular for $t = T_n = \sqrt{(m/k)}n\pi$ in the one-dimensional harmonic oscillator problem, which means that given initial position Q and final position $q(T_n)$, it is impossible to determine uniquely the associated initial and final momenta, P and $p(T_n)$. On the other hand, if we are given an initial momentum P and final position $q(T_n)$, then we can uniquely find the corresponding initial position Q and final momentum $p(T_n)$ from the F_2 function. The initial and final positions are not independent variables at initial time, whereas q and P are. In the development of Hamilton–Jacobi theory, the importance of independent variables in the generating functions is key because Eq. (10) does not yield Eqs. (11–13) when the variables are not independent.

When the boundary value problem is ill-posed, the coefficients of the generating functions "blow up." Note that the generating function may have a limit as t goes to the singular time even if the coefficients blow up. For instance, in the case of the harmonic oscillator, F_1 behaves as $(q-Q)^2/[t-\sqrt{(m/k)}n\pi]$ as t goes to $\sqrt{(m/k)}n\pi$, but $q-Q$ goes to zero simultaneously. Hence, F_1 is defined in the limit, but the associated coefficient of the $(q-Q)^2$ term, $1/[t-\sqrt{(m/k)}n\pi]$, is not defined as t goes to $\sqrt{(m/k)}n\pi$. This can also be understood at a physical level. As the time interval between our initial and final locations becomes arbitrarily small, we require a larger and larger initial momentum to transfer a spacecraft from one position to the next in a finite time interval. In the limit, as t goes to zero, the momentum (velocity) required becomes infinite, unless the initial and final position are collocated, which, for us is a trivial solution that does not yield a unique set of momenta. Hence, if a multiple solution exists, the coefficient will become ill-defined and the numerical integration will stop.

To overcome such difficulties, we use the Legendre transformation [Eq. (15)]. The idea is as follows:

1) Integrate F_2 or F_3 until they become singular. (Let us call t_1 the singular time.)

2) Use the Legendre transformation to evaluate F_1 or F_4 at some epoch $t < t_1$. For polynomial generating functions, the Legendre transformation involves reversion of series.

3) Integrate F_1 or F_4 until they become singular at t_2. Because the generating functions are not all singular at the same time, $t_2 > t_1$ in general.

4) Use the Legendre transformation to find F_2 or F_3 at some time $t > t_1$.

5) Integrate F_2 or F_3 up to the next singularity and use the same technique again.

Application to Formation Flight Transfers

We have developed all of the theory and tools we need to study the reconfiguration of spacecraft formations using our approach. Let us recall the main issues:

1) The generating functions are found as a function of either $(\Delta q, \Delta Q, t)$, $(\Delta q, \Delta P, t)$, $(\Delta p, \Delta Q, t)$, or $(\Delta p, \Delta P, t)$, and they solve the relative boundary value problem, that is, the boundary value problem relative to a nominal trajectory.

2) We made the assumption that $(\Delta q, \Delta p)$ are small to ensure the convergence of the series in Eq. (50). Hence, we find local solutions to the Hamilton–Jacobi equation in the spatial domain, but global solutions in the time domain.

3) F_1 and F_4 are undefined at $t = 0$.

4) F_2 and F_3 are well-defined at $t = 0$.

5) It is possible to transform from one generating function to another by the use of the Legendre transformation in Eq. (15).

6) It can be proven that a maximal solution to the differential equations defining F_i exists, but there is no guarantee that the solution is defined for all time, that is, singularities may exist.

7) Numerical integration of Eq. (61) provides a solution at order two to the relative boundary value problem. To get an approximation of order k, we must keep k terms in the expression of H_h, which is best done by the use of a symbolic manipulator.

Given these points, we are able to compute the generating functions to high order in the spatial domains over arbitrary time intervals (except in the vicinity of singularities).

Example Problem

We consider the following problem: Consider a constellation of spacecraft located at the libration point L_2 of the Hill's three-body problem at $t = 0$ (the Hill's problem and equations are given in Ref. 17). Their application to spacecraft motion is discussed by Scheeres et al.[18] (also see Appendices A and B). At a later time $t = t_f$, we want the spacecraft to lie on a circle surrounding the libration point at a distance of 108,000 km (Fig. 1). What initial velocity is required to transfer to this circle in time t_f, and what will the final velocity be once we arrive at the circle? The answer will depend, of course, on where we arrive on the circle. In general, this problem must be solved repeatedly for each point on the circle we wish to transfer to and for each transfer time. In our example, we only need to compute the generating functions to high enough order to be able to compute the answer as an analytic function of the final location.

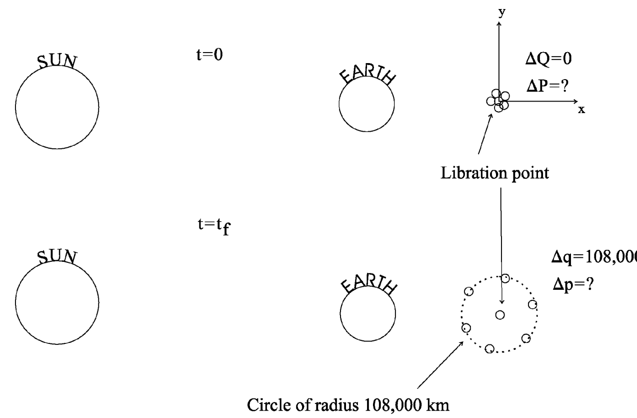

Fig. 1 Spacecraft formation at initial and final times.

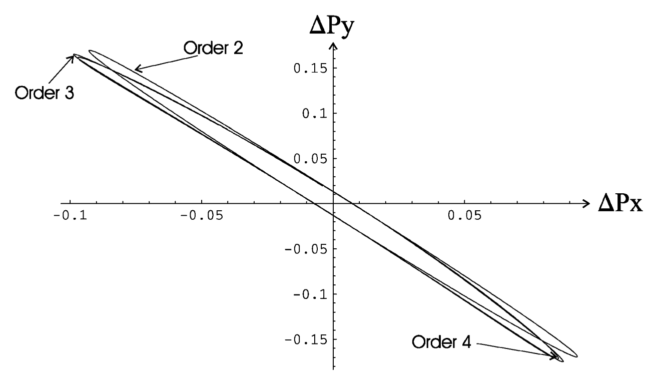

Fig. 2 ΔP in normalized units for $t = 82$ days, 1 unit $\leftrightarrow$ 432 m · s^{-1}.

To solve such a problem we must first find F_1. We proceed along the following steps: We solve the Hamilton–Jacobi equation for F_2 at order four for $0 \le t < t_s$, where t_s is the time at which F_2 becomes singular. Then we use the Legendre transformation to find the value of F_1 at $t_1 < t_s$ as a function of $F_2(t = t_1)$. We solve the Hamilton–Jacobi equation for F_1 for $t_0 \le t \le t_2$ where $t_2 > t_s$ and $t_0 \le t_1$. Again, we use Legendre transformation to find the value of F_2 at t_2, and we continue solving the Hamilton–Jacobi equation until we encounter another singularity. We repeat these steps to avoid each singularity. In this manner, we obtain F_1 at order four for every time except around the singularities of F_1.

Figure 2 represents the normalized initial momentum required to accomplish the maneuver in 82 days (ΔP_2 as a function of ΔP_1) solved at order two, three, and four. The coordinates of a point on the curves represented in Fig. 2 correspond to the value of the components of the initial momentum along the x and y axis, which allow the spacecraft to move from the the libration point to a distance of 108,000 km in 82 days. As expected, the second-order plot is a quadratic curve (only linear effects are taken into account), whereas the third- and fourth-order curves exhibit nonlinear effects.

We can verify the convergence and the accuracy of the solution just found by generating trajectories with initial conditions $\Delta Q = 0$ and the ΔP as found from the generating function, then comparing the final position with the expected one. Figure 3 represents the normalized error in final position. As we can see, the order two solution provides a poor approximation to the initial momentum because the error ranges up to 9615 km. Orders three and four give order of magnitude improvements because the error is less than 480 km for order three and less than 77 km for order four, more than two orders of magnitude better than the order two solution.

In this example, the transfer time is set and we considered the initial momentum only. For an actual reconfiguration, t_f is free, and we may wish to minimize the fuel spent. In such a case, we

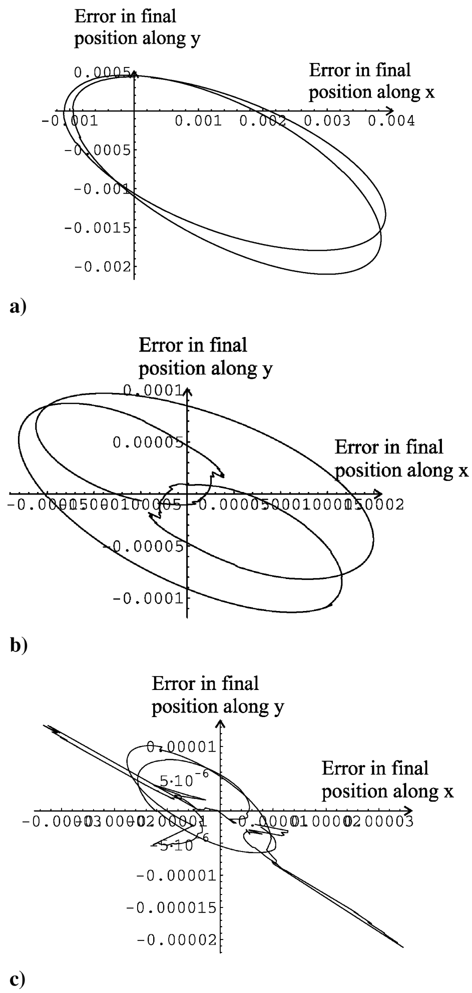

Fig. 3 Error in the normalized final position, 0.01 unit ↔ 21,500 km: a) order two, b) order three, and c) order four.

must take into account the value of the final impulse required to stop the spacecraft. Through this example, we see that, depending on the final position of the spacecraft on the circle, very different values of initial momenta may be required. (The norm of the initial momentum ranges from 0.01 to 0.1.)

We now allow the transfer time t_f to vary and then consider the full problem of optimal impulsive transfer for an arbitrary collection of spacecraft from the libration point to the circle. Note that because we have already solved the generating functions in the preceding example and stored them as functions of time, all of the following applications only require the evaluation of the polynomial functions $\partial F_1/\partial \Delta_q$ and $\partial F_1/\partial \Delta Q$.

Properties of L_2

We take the same transfer as before, but now vary the transfer time and look at the initial momentum as a function of time. Figure 4 shows the plot of the initial momentum for $t = 6$ days, until $t = 47$ days, and Fig. 5 represents the same for $t \in [41$ days, 88 days$]$. Each point along the curve will transfer to a different location on the circle in the specified transfer time. We notice that for small enough t, the portrait of the initial momentum is a circle. This implies that whichever direction the spacecraft leaves L_2, if its initial momentum has the given magnitude, then it will be at a distance of 108,000 km from L_2 at t. In this case, the time is short and the speeds large enough so that the motion is essentially rectilinear. For large enough t, the phase portrait is an ellipse, implying that the system is controlled by the quadratic terms of F_1. The semiminor axis indicates the direction along which the initial momentum is minimal, whereas the semimajor axis indicates the direction along which the initial momentum is maximal. These two directions correspond to the unstable and stable manifolds of L_2. Along the unstable manifold, the spacecraft naturally speeds up and, thus, requires a smaller initial momentum, whereas along the stable manifold it slows down and, thus, requires a larger initial momentum. Moreover, because we take nonlinear terms into account, Fig. 5 provides a better estimate of the direction of the stable manifold than the one usually found by the use of the eigenvectors of the linearized system.

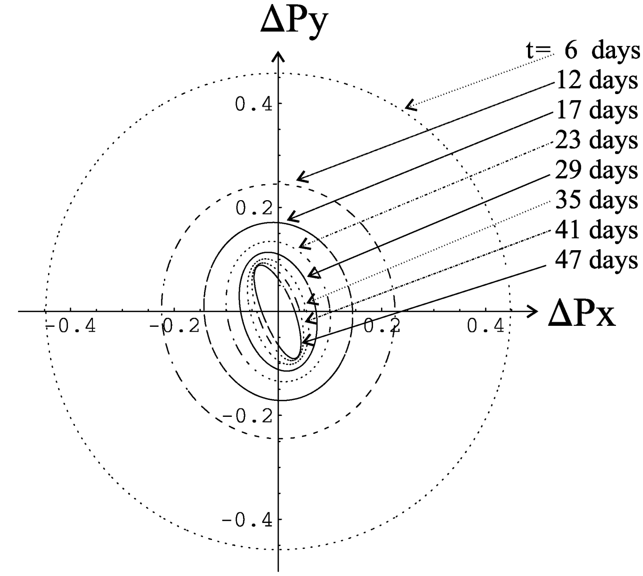

Fig. 4 ΔP in normalized units for $t \in 2\,(6$ days, 47 days$)$, 1 unit ↔ 432 m · s^{-1}.

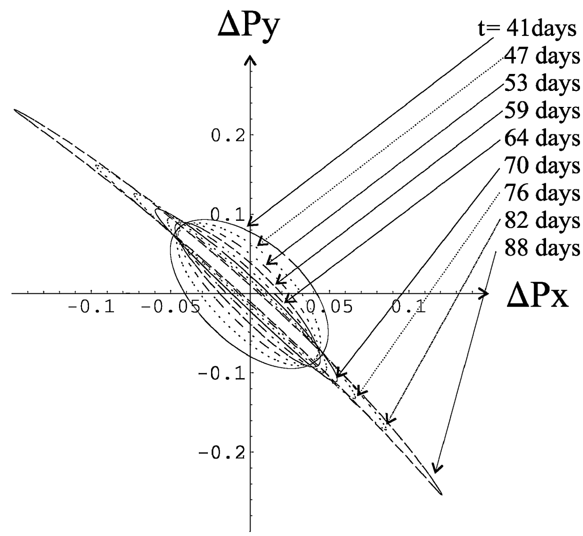

Fig. 5 ΔP in normalized units for $t \in (41$ days, 88 days$)$, 1 unit ↔ 432 m · s^{-1}.

Additionally, we can see the effects of the characteristic time of the equilibrium point by observing the change in evolution of the curve. By definition, the characteristic time is the minimum amount of time necessary for the spacecraft to be influenced by the properties of the equilibrium point, namely, the unstable and stable manifolds. Using Fig. 4, we see that the spacecraft is under the influence of the stable and unstable manifolds when the phase portrait of the initial momentum deviates markedly from a circle, which occurs at $t_c = 29$ days. Thus, for $t \leq t_c$ the instability properties of L_2 are not important. Hence, in the preceding example, if the time of transfer is chosen to be less than t_c, any position on the circle requires the same initial momentum in magnitude; that is, there are no privileged locations. For longer times, $t > t_c$, the geometry of the libration point plays an important role. To conclude, we notice that as t increases, the norm of the initial momentum needed to move the spacecraft to the circle decreases. However, for transfer times longer than 50 days, some positions on the circle require less and less initial momentum, but others require a larger and larger initial momentum. More precisely,

1) If the transfer time is less than 50 days, the norm of the initial momentum needed decreases as time increases. Hence, to achieve a transfer in a time $t_f \leq 50$ days, it is best to do it in time t_f, whatever the final position of the spacecraft on the circle.

2) If the time transfer is more than 50 days, the situation becomes more complicated. There are positions on the final circle for which large transfer times minimize the initial momentum required. These positions correspond to those close to the unstable manifold. On the other hand, if the spacecraft is leaving along the stable manifold, a transfer in about 50 days requires the minimum initial momentum.

Hence, if our goal is to minimize the initial momentum and achieve a transfer to the circle in less than 80 days, it is better to vary transfer times from 50 to 80 days, depending on what the positions of the spacecraft are on the circle. Thus, even for such a simple geometry, we see that there can be large changes in the "optimal" transfer as a function of time and geometry. Our method expresses all of this complex information within a series of polynomial coefficients tabulated as a function of time. Furthermore, we can repeat similar exercises for arbitrary initial and final positions relative to the libration point.

Optimization

In this section, we consider the same problem, but now we wish to minimize the total fuel cost of the maneuver, that is, to minimize the sum of the norm of the initial momentum and the norm of the final momentum, $\sqrt{(|\Delta P|^2 + |\Delta p|^2)}$. We assume zero momentum in the Hill's rotating frame at the beginning and end of the maneuver. Although not a realistic maneuver, we can use it to exhibit the applicability of our approach.

Figures 6–8 show the value of $\sqrt{(|\Delta P|^2 + |\Delta p|^2)}$ as a function of position in the final formation at different times. Define the final position of the spacecraft as $\Delta \boldsymbol{q} = \Delta q \hat{\boldsymbol{q}}$, where $\Delta q = 108{,}000$ km and $\hat{\boldsymbol{q}}$ is the unit vector pointing toward the location of the final circle. Then, Figs. 6–8 represent $\sqrt{(|\Delta P|^2 + |\Delta p|^2 \hat{\boldsymbol{q}})}$. We notice three tendencies:

1) For t less than the characteristic time, no matter which direction the spacecraft leaves L_2, it costs essentially the same amount of fuel to reach the final position and stop (Fig. 6).

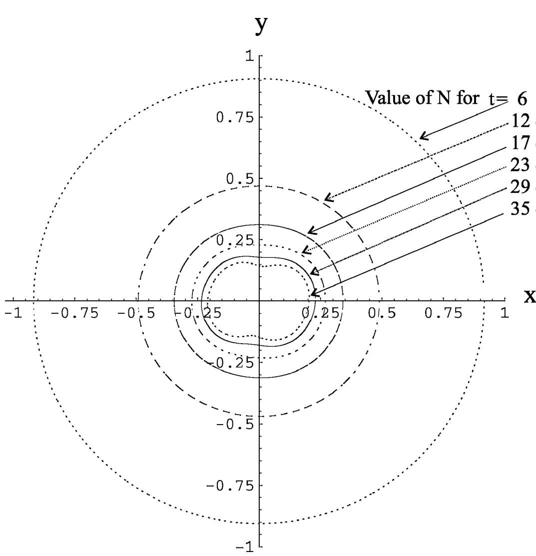

Fig. 6 $\sqrt{(|\Delta \boldsymbol{P}|^2 + |\Delta \boldsymbol{p}|^2)}\, \hat{q}$ **for** $t \in$ **[6 days, 35 days], 1 unit** $\leftrightarrow$ **432 m · s**$^{-1}$**.**

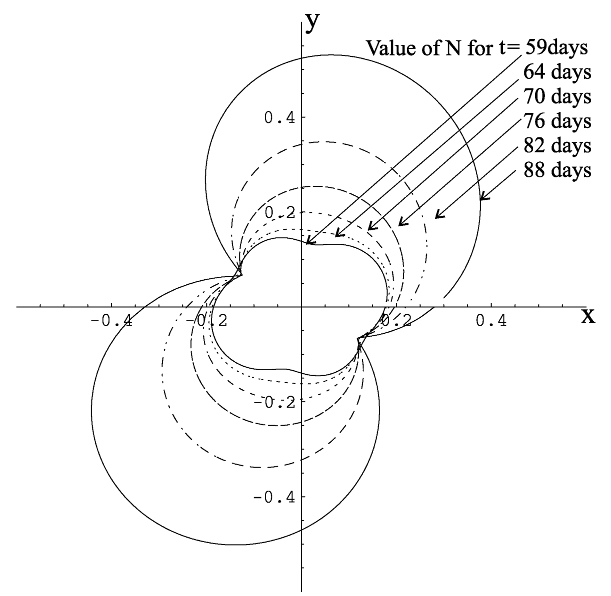

Fig. 8 $\sqrt{(|\Delta \boldsymbol{P}|^2 + |\Delta \boldsymbol{p}|^2)}\, \hat{q}$ **for** $t \in$ **[59 days, 88 days], 1 unit** $\leftrightarrow$ **432 m · s**$^{-1}$**.**

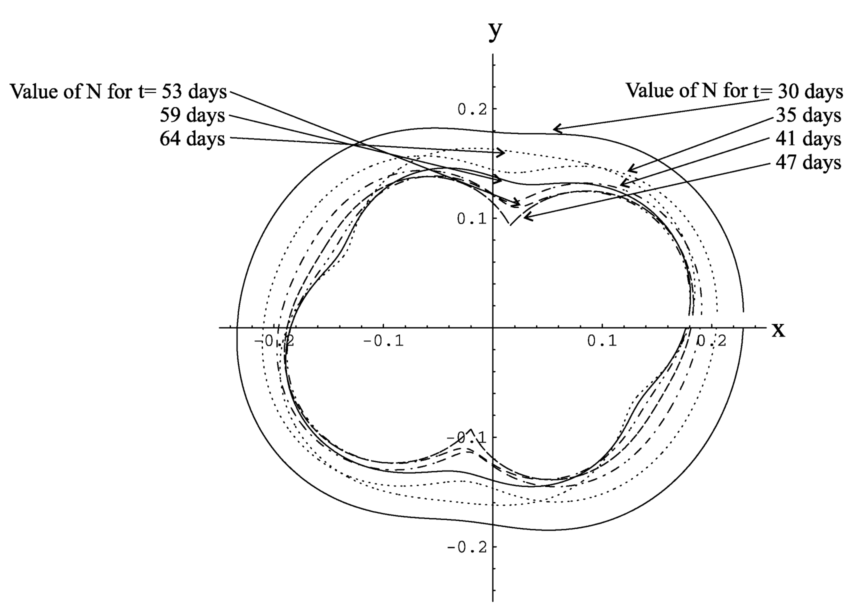

Fig. 7 $\sqrt{(|\Delta \boldsymbol{P}|^2 + |\Delta \boldsymbol{p}|^2)}\, \hat{q}$ **for** $t \in$ **[30 days, 64 days], 1 unit** $\leftrightarrow$ **432 m · s**$^{-1}$**.**

2) For t larger than the characteristic time, but less than 47 days, the curve describing $\sqrt{(|\Delta P|^2 + |\Delta p|^2)}$ shrinks along a direction 80 deg from the x direction. Thus, placing a spacecraft on the final circle at an angle of 80 or 260 deg from the x axis provides the lowest cost in fuel (Fig. 7).

3) For t larger than 47 days, the curve describing $\sqrt{(|\Delta P|^2 + |\Delta p|^2)}$ shrinks along a direction perpendicular to the preceding one, at an angle of $\sim$170 deg with the x axis, and expands along the 80-deg direction. Thus, there exists an epoch for which placement of a spacecraft on the final circle at an angle of 170 or 350 deg from the x axis provides the lowest final cost. This happens for $t = 88$ days (Figs. 7 and 8).

To conclude, we see that the optimal transfer time to the final circle changes as a function of location on the circle. Whereas this is to be expected, our results provide direct solutions for this nonlinear boundary value problem. Moreover, the solution can be adapted to arbitrary relative motions about the libration points.

We now make a few additional remarks to emphasize the advantage of our method. First, additional spacecraft do not require any additional computations. Hence, our method to design optimal reconfiguration is valid for infinitely many spacecraft in formation. Second, now that we have computed the generating functions around the libration point, we are able to analyze any reconfiguration around the libration point at the cost of evaluating a polynomial function. Furthermore, our algorithm could have been applied in the same way and at the same cost in terms of computation to any other gravitational field, including the influence of a noncircular orbit (elliptic restricted three-body problem), influence of a fourth body, etc. Finally, if the formation of spacecraft is evolving around a base that is on a given trajectory, we can linearize about this trajectory and then proceed as in the preceding examples to study the reconfiguration problem.

Conclusions

This paper derives a novel application of the Hamilton–Jacobi equation to solve the problem of nonlinear targeting for a formation of spacecraft. The application constructs direct solutions for the Taylor series expansion of the generating function to arbitrarily high order in displacements from a nominal, numerically defined trajectory. By extending the procedure to high degree in powers of the specified boundary variables, we are able to immediately generate nonlinear solutions to boundary value problems. By formulating the problem as a set of ordinary differential equations, we are able to generate time histories of the boundary value problem as well. In this paper, we demonstrate the theory and provide examples that show its use for the design and optimization of the nonlinear relative motion of a formation of spacecraft. Through the reconfiguration problem about the libration point in the Hill's three-body problem, we emphasize the main advantages of the use of the generating functions. Specifically, we have pointed out that we can deal with any number of spacecraft flying in any gravitational field and that once the generating functions are known in closed form, we are free to change the shape of the formation. Thus, although we specifically solved the deployment problem around the libration point, the same generating function could be used to study a reconfiguration problem around the libration point from an arbitrary configuration to another one at no additional cost except for the evaluation of a polynomial function. Finally, for these two impulsive transfers, the problem of optimization of a transfer is reduced to the simple task of reading a plot.

Appendix A: Hamilton–Jacobi Equations at Third Order

Define the following variables:

$$A_{i,j,k} = h_{n+a,n+b,n+c}(f_{a,i} + f_{i,a})(f_{b,j} + f_{j,b})(f_{c,k} + f_{k,c})$$

$$B_{i,j,k} = h_{n+a,n+b}(f_{a,i} + f_{i,a})(f_{b,j,k} + f_{j,k,b} + f_{k,b,j})$$

$$C_{i,j,k} = h_{n+a,n+b}(f_{b,i} + f_{i,b})(f_{a,j,k} + f_{j,k,a} + f_{k,a,j})$$

$$D_{a,i,j} = (h_{a,n+b,n+c} + h_{n+c,a,n+b} + h_{n+b,n+c,a}) \times (f_{b,i} + f_{i,b})(f_{c,j} + f_{j,c})$$

$$E_{a,i,j} = (h_{a,n+b} + h_{n+b,a})(f_{b,i,j} + f_{j,b,i} + f_{i,j,b})$$

$$G_{a,b,i} = (h_{a,b,n+c} + h_{b,n+c,a} + h_{n+c,a,b})(f_{c,i} + f_{i,c}) \tag{A1}$$

Keeping only third-order terms in Eq. (71) yields

$$\dot{f}_{i,j,k} y_i y_j y_k + (A_{i,j,k} + B_{i,j,k} + C_{i,j,k}) y_i y_j y_k + (D_{a,i,j} + E_{a,i,j}) y_a y_i y_j + G_{a,b,i} y_a y_b y_i + h_{a,b,c} y_a y_b y_c = 0 \tag{A2}$$

We deduce the coefficients of $y_i y_j y_k$ as follows:

1) The coefficients of $y^3_{i\le n}$ are

$$A_{i,i,i} + B_{i,i,i} + C_{i,i,i} + D_{i,i,i} + E_{i,i,i} + \dot{f}_{i,i,i} + G_{i,i,i} + h_{i,i,i} = 0 \tag{A3}$$

2) The coefficients of $y^3_{i>n}$ are

$$A_{i,i,i} + B_{i,i,i} + C_{i,i,i} + \dot{f}_{i,i,i} = 0 \tag{A4}$$

3) The coefficients of $y^2_{i\le n} y_{j\le n}$ are

$$(A + B + C + D + E + \dot{f} + G + h)_{\tau(i,i,j)} = 0 \tag{A5}$$

where $\tau(i, j, k)$ represents all of the distinct permutations of (i, j, k), that is, $A_{\tau(i,j,k),l} = A_{i,j,k,l} + A_{i,k,j,l} + A_{k,i,j,l} + A_{k,j,i,l} + A_{j,k,i,l} + A_{j,i,k,l}$, but $A_{\tau(i,i,j),l} = A_{i,i,j,l} + A_{i,j,i,l} + A_{j,i,i,l}$.

4) The coefficients of $y^2_{i\le n} y_{j>n}$ are

$$(A + B + C + \dot{f})_{\tau(i,i,j)} + (D + E)_{i,\tau(i,j)} + G_{i,i,j} = 0 \tag{A6}$$

5) The coefficients of $y_{i\le n} y_{j\le n} y_{k\le n}$ are

$$(A + B + C + D + E + \dot{f} + G + h)_{\tau(i,j,k)} = 0 \tag{A7}$$

6) The coefficients of $y_{i\le n} y_{j\le n} y_{k>n}$ are

$$(A + B + C + \dot{f})_{\tau(i,j,k)} + (D + E)_{i,\tau(j,k)} + (D + E)_{j,\tau(i,k)} + G_{\tau(i,j),k} = 0 \tag{A8}$$

7) The coefficients of $y^2_{i>n} y_{j\le n}$ are

$$(A + B + C + \dot{f})_{\tau(i,i,j)} + (E + D)_{j,i,i} = 0 \tag{A9}$$

8) The coefficients of $y^2_{i>n} y_{j>n}$ are

$$(A + B + C + \dot{f})_{\tau(i,i,j)} = 0 \tag{A10}$$

9) The coefficients of $y_{i\le n} y_{j>n} y_{k>n}$ are

$$(A + B + C + \dot{f})_{\tau(i,j,k)} + (D + E)_{i,\tau(j,k)} = 0 \tag{A11}$$

10) The coefficients of $y_{i>n} y_{j>n} y_{k>n}$ are

$$(A + B + C + \dot{f})_{\tau(i,j,k)} = 0 \tag{A12}$$

Equations (A3–A12) allow us to solve for F_2 or F_1.

Appendix B: Hill's Problem

Hill's problem is a three-body problem where the first body has a larger mass than the second one and the third one has negligible mass.[17] The coordinate system is centered on the second body and the equations are motion are

$$\ddot{x} - 2\dot{y} = -x/r^3 + 3x \tag{B1}$$

$$\ddot{y} + 2\dot{x} = -y/r^3 \tag{B2}$$

$$\ddot{x} = -z/r^3 - z \tag{B3}$$

where $r^2 = x^2 + y^2 + z^2$.

In the present study, we consider the two-dimensional problem, where z is set to 0. The Lagrangian then reads

$$L(q,\dot{q},t) = \tfrac{1}{2}\left(\dot{q}_x^2 + \dot{q}_y^2\right) + 1\Big/\sqrt{q_x^2+q_y^2} + \tfrac{3}{2}q_x^2 - (\dot{q}_x q_y - \dot{q}_y q_x) \tag{B4}$$

Hence,

$$p_x = \frac{\partial L}{\partial \dot{q}_x} = \dot{q}_x - q_y \tag{B5}$$

$$p_y = \frac{\partial L}{\partial \dot{q}_y} = \dot{q}_y + q_x \tag{B6}$$

From Eqs. (B4–B6), we obtain the Hamiltonian function H:

$$H(q,p) = p_x\dot{q}_x + p_y\dot{q}_y - L = \tfrac{1}{2}\left(p_x^2+p_y^2\right) + (q_y p_x - q_x p_y) - 1\Big/\sqrt{q_x^2+q_y^2} + \tfrac{1}{2}\left(q_y^2 - 2q_x^2\right) \tag{B7}$$

There are two equilibrium points for this system that are called libration points. Their coordinates are $L_1[-(\frac{1}{3})^{1/3},0]$ and $L_2[(\frac{1}{3})^{1/3},0]$.

In this paper, we study a reconfiguration problem about the libration point L_2 in the Hill's problem. Hence, we need to describe the relative motion with respect to L_2, that is, the nominal trajectory reduces to a point. From Eqs. (B5) and (B6)

$$q_0(t) = \left[\left(\tfrac{1}{3}\right)^{\frac{1}{3}}, 0\right], \qquad p_0(t) = \left[0, \left(\tfrac{1}{3}\right)^{\frac{1}{3}}\right] \tag{B8}$$

Now, by the use of Eq. (55), we can compute H_h, the Hamiltonian function describing the relative motion:

$$H_h = \frac{1}{2}X_h^T \begin{pmatrix} H_{qq}(t) & H_{qp}(t) \\ H_{pq}(t) & H_{pp}(t)\end{pmatrix} X_h + \cdots \tag{B9}$$

where

$$X_h = \begin{pmatrix}\Delta q_x \\ \Delta q_y \\ \Delta p_x \\ \Delta p_y\end{pmatrix}$$

$$H_{qq}(t) = \begin{pmatrix} \dfrac{1}{\left(q_{0x}^2+q_{0y}^2\right)^{\frac{3}{2}}} - \dfrac{3q_{0x}^2}{\left(q_{0x}^2+q_{0y}^2\right)^{\frac{5}{2}}} - 2 & -\dfrac{3q_{0x}q_{0y}}{\left(q_{0x}^2+q_{0y}^2\right)^{\frac{5}{2}}} \\ -\dfrac{3q_{0x}q_{0y}}{\left(q_{0x}^2+q_{0y}^2\right)^{\frac{5}{2}}} & \dfrac{1}{\left(q_{0x}^2+q_{0y}^2\right)^{\frac{3}{2}}} - \dfrac{3q_{0y}^2}{\left(q_{0x}^2+q_{0y}^2\right)^{\frac{5}{2}}} + 1 \end{pmatrix} \tag{B10}$$

$$H_{qp}(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{B11}$$

$$H_{pq}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{B12}$$

$$H_{pp}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{B13}$$

Replacement by the values of $[q_{0x}(t), q_{0y}(t), p_{0x}(t), p_{0y}(t)] = [(\frac{1}{3})^{1/3}, 0, 0, (\frac{1}{3})^{1/3}]$ yields the expression of H_h at second order:

$$H_h = \frac{1}{2}(\Delta q_x \quad \Delta q_y \quad \Delta p_x \quad \Delta p_y)\begin{pmatrix} -8 & 0 & 0 & -1 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}\Delta q_x \\ \Delta q_y \\ \Delta p_x \\ \Delta p_y\end{pmatrix} \tag{B14}$$

At higher order, we find

$$H_h = \frac{1}{2}(\Delta q_x \quad \Delta q_y \quad \Delta p_x \quad \Delta p_y)\begin{pmatrix} -8 & 0 & 0 & -1 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}\Delta q_x \\ \Delta q_y \\ \Delta p_x \\ \Delta p_y\end{pmatrix} + 3^{\frac{4}{3}}\Delta q_x^3 - \frac{3^{\frac{7}{3}}}{2}\Delta q_x \Delta q_y^2 - 3^{\frac{5}{3}}\Delta q_x^4 + 3^{\frac{8}{3}}\Delta q_x^2 \Delta q_y^2 - \frac{3^{\frac{8}{3}}}{8}\Delta q_y^4 \ldots \tag{B15}$$

Note that the Hamiltonian function describing the dynamics of a particle relative to a nominal trajectory that reduces to an equilibrium point is time independent.

Acknowledgment

This work was funded in part by NASA's Office of Space Science and by the Interplanetary Network Technology Program by a grant from the Jet Propulsion Laboratory, California Institute of Technology, which is under contract with NASA.

References

[1]Zhan, H., Xu, B., and Zhang, Z. M., "An Analytical Approach to Analyzing Spacecraft Formation-Flying Motion," AIAA Paper 2002-4847, Aug. 2002.

[2]Richardson, D. L., and Mitchell, J. W., "A Third-Order Analytical Solution for Relative Motion with a Circular Reference Orbit," *AAS/AIAA Space Flight Mechanics Meeting*, Vol. 112, Pt. 1, American Astronautical Society, Springfield, VA, 2002, pp. 605–615.

[3]Alfriend, K. T., Yan, H., and Valadi, S. R., "Nonlinear Considerations in Satellite Formation Flying," AIAA Paper 2002-4741, Aug. 2002.

[4]Wang, P. K. C., and Hadaegh, F. Y., "Minimum-Fuel Formation Reconfiguration of Multiple Free-Flying Spacecraft," *Journal of the Astronautical Sciences*, Vol. 47, Nos. 1–2, 1999, pp. 77–102.

[5]Goldstein, H., *Classical Mechanics*, Addison Wesley Longman, Reading, MA, 1965.

[6]Greenwood, D. T., *Classical Dynamics*, Prentice–Hall, Upper Saddle River, NJ, 1977.

[7]Channell, P. J., and Scovel, J. C., "Symplectic Integration of Hamiltonian Systems," *Nonlinearity*, Vol. 3, No. 2, 1990, pp. 231–259.

[8]Abraham, R., and Marsden, J. E., *Foundations of Mechanics*, 2nd ed., W. A. Benjamin, Reading, MA, 1978.

[9]Arnold, V. I., *Mathematical Methods of Classical Mechanics*, 2nd ed., Springer-Verlag, Berlin, 1988, p. 267.

[10]Guibout, V. M., and Scheeres, D. J., "Formation Flight with Generating Functions: Solving the Relative Boundary Value Problem," AIAA Paper 2002-4639, Aug. 2002.

[11]Lanczos, C., *The Variational Principles of Mechanics*, 4th ed., Univ. of Toronto Press, Toronto, 1977.

[12]Marsden, J. E., and Ratiu, T. S., *Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems*, 2nd ed., Springer-Verlag, Berlin, 1998.

[13]Battin, R. H., *An Introduction to the Mathematics and Methods of Astrodynamics*, rev. ed., AIAA, Reston, VA, 1999.

[14]Arnold, V. I., *Catastrophe Theory*, 3rd rev. and expanded ed., Springer-Verlag, Berlin, 1992.

[15]Ehlers, J., and Newman, E. T., "The Theory of Caustics and Wavefront Singularities with Physical Applications," *Jounral of Mathematical Physics A*, Vol. 41, No. 6, 2000, pp. 3344–3378.

[16]Hörmander, L., *Seminar on Singularities of Solutions of Linear Partial Direrential Equations*, Princeton Univ. Press, Princeton, NJ, 1979.

[17]Arnold, V. I., Kozlov, V. V., and Neishtadt, A. I., *Mathematical Aspects of Classical and Celestial Mechanics, Dynamical Systems III*, Springer-Verlag, Berlin, 1988.

[18]Scheeres, D. J., Han, D., and Hou, Y., "Influence of Unstable Manifolds on Orbit Uncertainty," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 3, 2004, pp. 573–585.